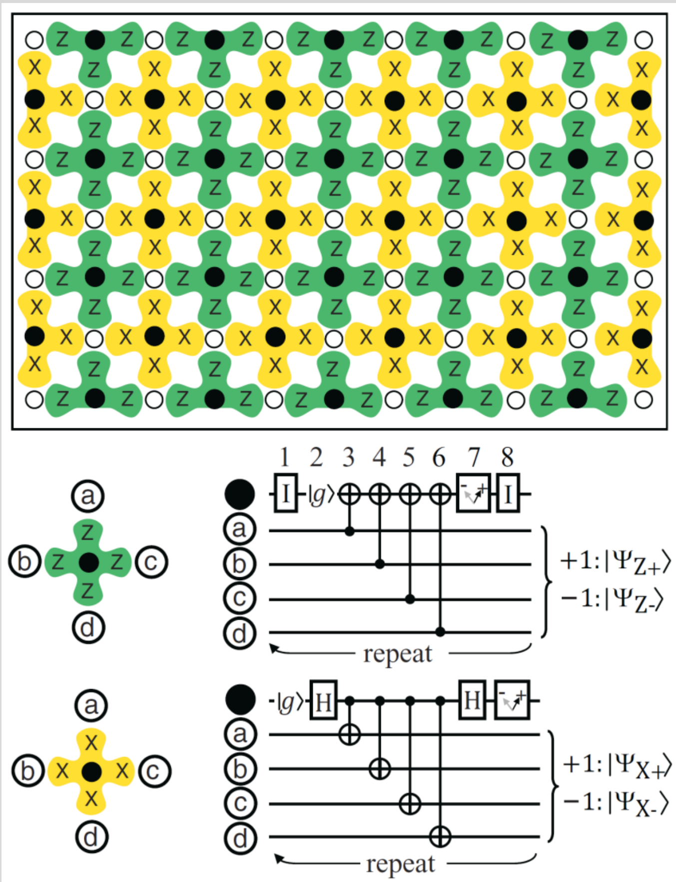


A (very) Brief Intro to Quantum Error Correction

ECE 592/CSC 591 – Fall 2018

NC STATE UNIVERSITY

Electrical & Computer Engineering

@NCStateECE

Agenda

- Errors -- where they come from, what they look like
- Replication-based codes
- Stabilizer codes and transversal operators
- Surface Codes

Errors

- Sources of errors
  - Control errors -- gates that are incorrectly applied
    - Usually based on faulty knowledge of the physical system characteristics
  - Environmental errors
  - Initialization, measurement, loss, leakage (into other energy states)

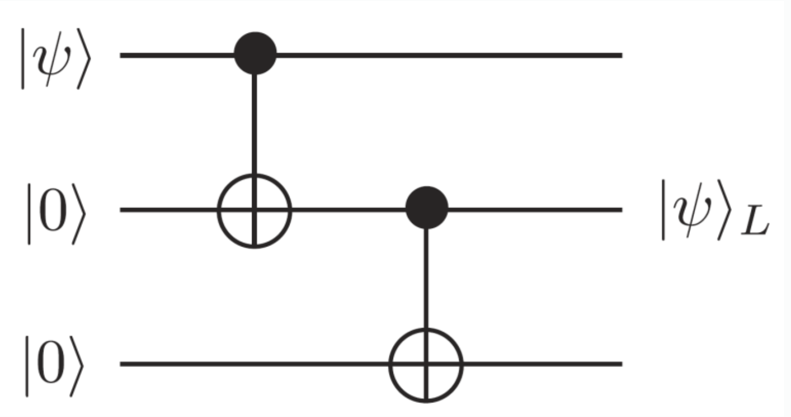
- What they look like
  - Random qubit flip (X) and phase change (Z)
  - Small errors (rotations) accumulate into observable flip/phase changes

3-qubit code: starting point…

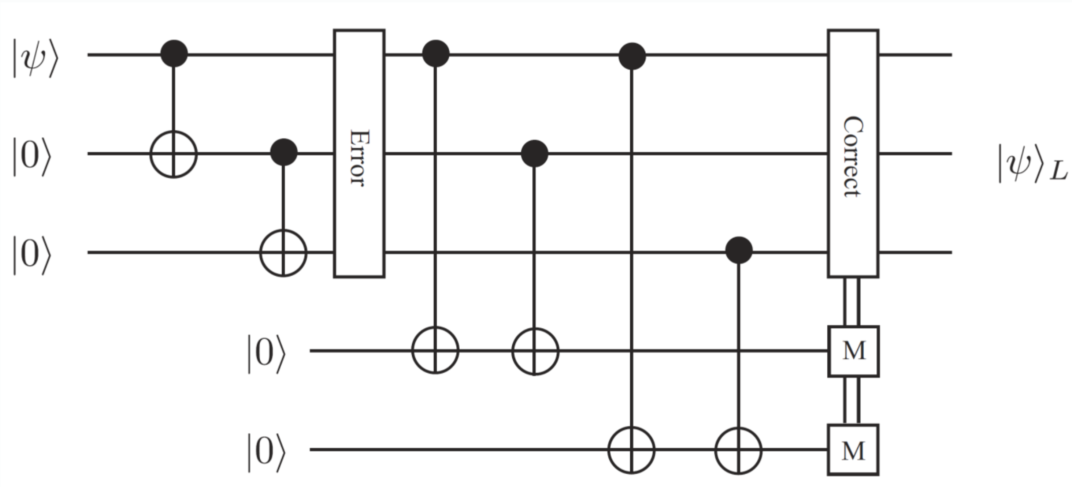
- Three physical qubits = 1 logical qubit
- Corrects a single flip error
  - Not a full quantum code, does not address phase error

$$|0\rangle_L = |000\rangle \,, \qquad |1\rangle_L = |111\rangle$$

$$\begin{aligned} \alpha\,|0\rangle + \beta\,|1\rangle &\to \alpha\,|0\rangle_L + \beta\,|1\rangle_L \\ &= \alpha\,|000\rangle + \beta\,|111\rangle \\ &= |\psi\rangle_L \,. \end{aligned}$$



Correcting a flip error



First ancilla checks whether first two qubits are equal.

Second ancilla checks whether first and third qubits are equal.

Result uniquely identifies the flip.
Apply X to error bit to correct.

Error Location	Final State, $\|\text{data}\rangle\,\|\text{ancilla}\rangle$
No Error	$\alpha\|000\rangle\|00\rangle + \beta\|111\rangle\|00\rangle$
Qubit 1	$\alpha\|100\rangle\|11\rangle + \beta\|011\rangle\|11\rangle$
Qubit 2	$\alpha\|010\rangle\|10\rangle + \beta\|101\rangle\|10\rangle$
Qubit 3	$\alpha\|001\rangle\|01\rangle + \beta\|110\rangle\|01\rangle$

Ancilla Measurement	Collapsed State	Consequence
00	$\alpha\|000\rangle + \beta\|111\rangle$	No Error
01	$\alpha\|001\rangle + \beta\|110\rangle$	σ_x on Qubit 3
10	$\alpha\|010\rangle + \beta\|101\rangle$	σ_x on Qubit 2
11	$\alpha\|100\rangle + \beta\|011\rangle$	σ_x on Qubit 1

Multi-qubit error?

Error Location	Final State, $\lvert\text{data}\rangle \lvert\text{ancilla}\rangle$	Assumed Error
Qubit 1 & 2	$\alpha \lvert 110\rangle \lvert 01\rangle + \beta \lvert 001\rangle \lvert 01\rangle$	σ_x on Qubit 3
Qubit 2 & 3	$\alpha \lvert 011\rangle \lvert 11\rangle + \beta \lvert 100\rangle \lvert 11\rangle$	σ_x on Qubit 1
Qubit 1 & 3	$\alpha \lvert 101\rangle \lvert 10\rangle + \beta \lvert 010\rangle \lvert 10\rangle$	σ_x on Qubit 2
Qubit 1, 2 & 3	$\alpha \lvert 111\rangle \lvert 00\rangle + \beta \lvert 000\rangle \lvert 00\rangle$	No Error

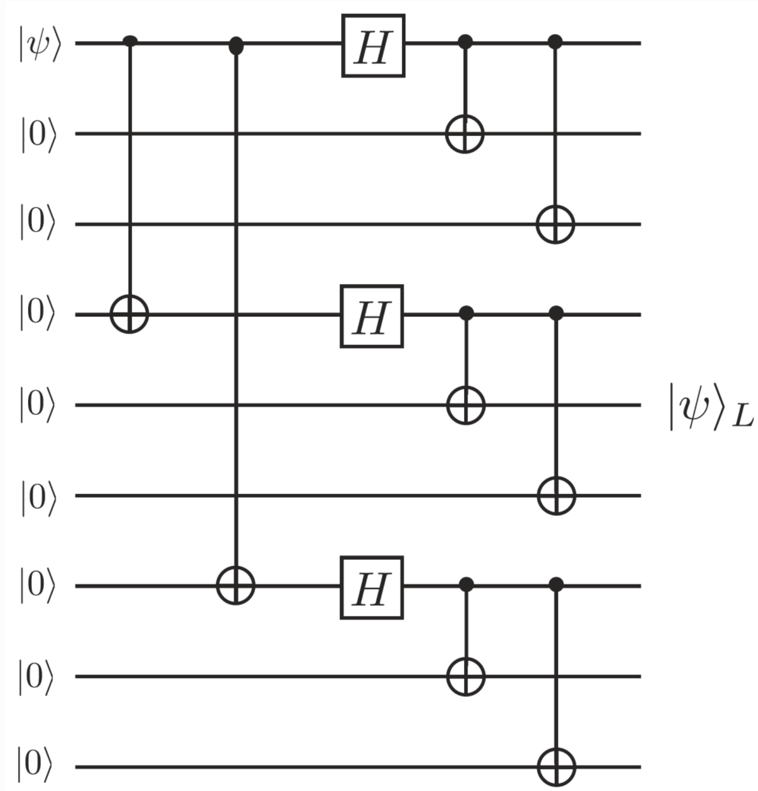
Multi-qubit error is incorrectly identified as a single-qubit error, and the wrong correction is applied. (Creates a valid state, but not the correct state.)

9-qubit Shor code

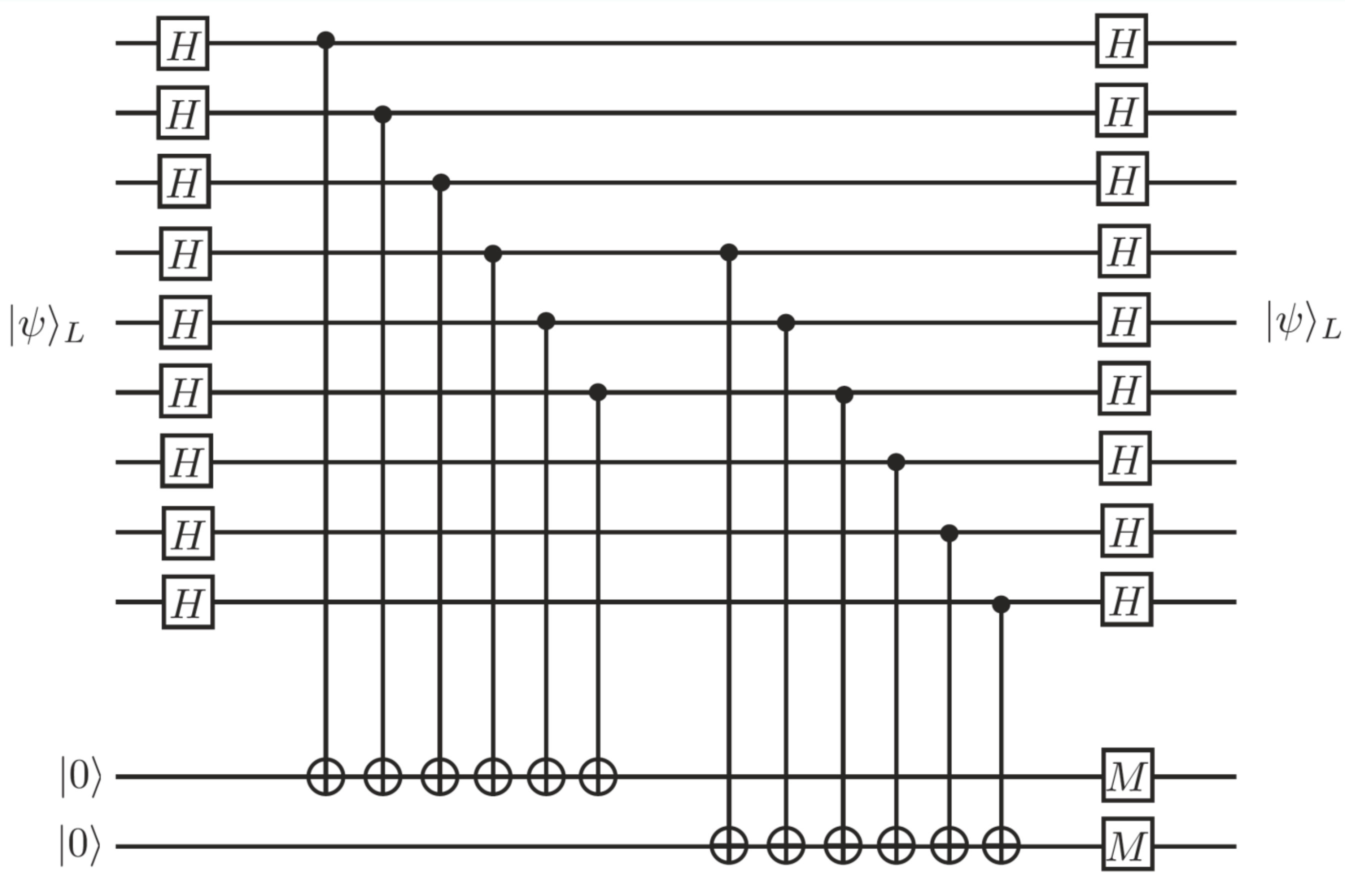
- Corrects a single flip (X) or phase (Z) error, or both

$$|0\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Flip correction = same circuit as before on each group of three.



Correcting a phase error



First ancilla checks whether sign of first three equals sign of second three.

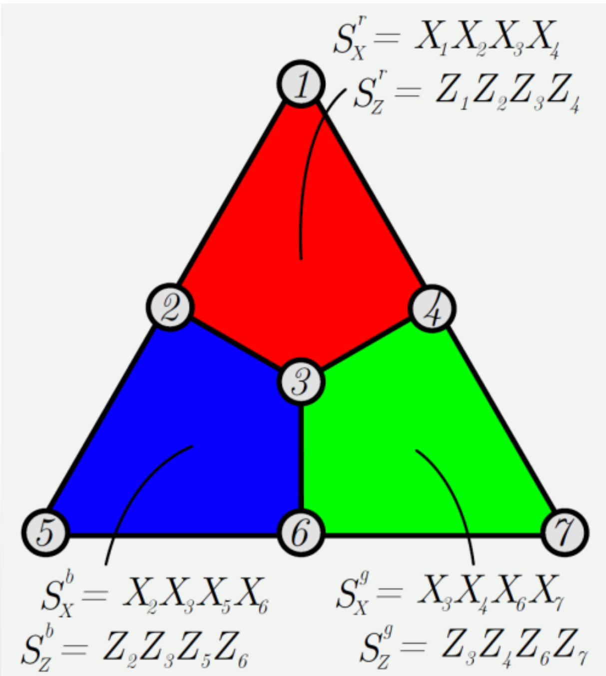
Second ancilla checks sign of 2nd and 3rd groups.

Syndome bits identify which group of three; correction applied to ANY qubit in the group.

https://goo.gl/BkhgWk

Summary of 9-qubit Shor Code

- Correct a single flip error in any of the nine qubits
  - Actually, can correct one qubit in all three groups, but this is considered a single-qubit code because not all multi-flip errors can be corrected.
- Correct a single phase error in any of the nine qubits
  - Correct with a Z on any qubit of the group
- If both X and Z error occurs, both will be corrected
  - Even if on the same qubit

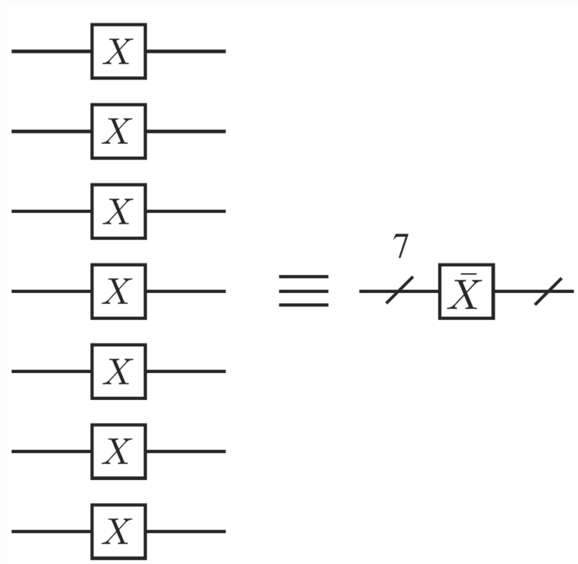


Stabilizers & Transversal Gates

Two important concepts

- Stabilizer codes
  - Systematic way to generate codes and correction circuits
  - Code words based on +1 and -1 eigenstates for a particular set of operators known as **stabilizers**

- Transversal gates
  - If an operator is transversal, then applying the operator (gate) to an encoded state is a matter of applying the same (or equally simple) gate to each individual bit
  - This means that computations can be just as efficient with encoded states



Stabilizers

- A state psi is **stabilized** by an operator K if it has eigenvalue +1: $K|\psi\rangle = |\psi\rangle$
- An N-qubit stabilizer state $|\psi\rangle_N$ is defined by the N generators of a particular subset of the N-qubit Pauli operators (details in paper). The state is stabilized by all operators in the subset.

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}},$$
$$|\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}},$$

$$\Phi^+ \equiv \begin{pmatrix} K^1 = XX \\ K^2 = ZZ \end{pmatrix} \quad \Phi^- \equiv \begin{pmatrix} K^1 = -XX \\ K^2 = ZZ \end{pmatrix}$$
$$\Psi^+ \equiv \begin{pmatrix} K^1 = XX \\ K^2 = -ZZ \end{pmatrix} \quad \Psi^- \equiv \begin{pmatrix} K^1 = -XX \\ K^2 = -ZZ \end{pmatrix}$$

Stabilizer Codes

- Reduce N-qubit Hilbert space by requiring all codewords to be stabilized by a set of operators.
  - Effectively, (N-1) operators reduces N qubits to a 1-qubit logical space.
  - Example, given a 2-qubit space, require that each state be stabilized by XX. Only two orthogonal basis states satisfy this requirement:

$$|0\rangle_L \equiv \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \quad |1\rangle_L \equiv \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

- For QEC, stabilizer codes are design to support detection and correction.

7-qubit Steane Code

$$|0\rangle_L = \frac{1}{\sqrt{8}}(\,|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$$

$$|1\rangle_L = \frac{1}{\sqrt{8}}(\,|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle)$$

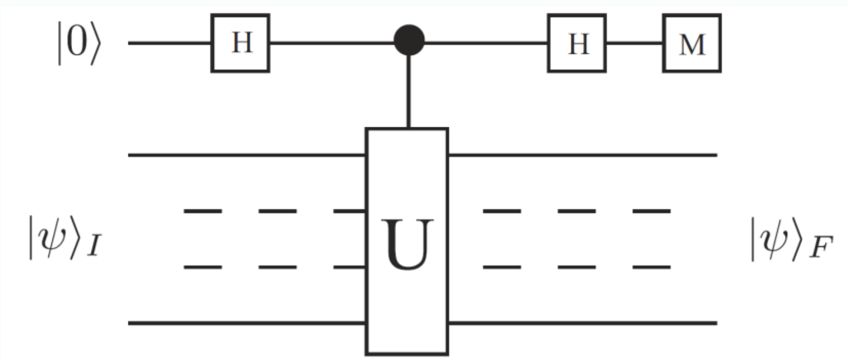
$$\begin{array}{ll} K^1 = IIIXXXX, & K^2 = XIXIXIX, \\ K^3 = IXXIIXX, & K^4 = IIIZZZZ \\ K^5 = ZIZIZIZ, & K^6 = IZZIIZZ. \end{array}$$

Stabilizers

Known as a [[7,1,3]] code
- 7 physical qubits
- 1 logical qubit
- distance 3 between states
- corrects (3-1)/2 = 1 error

Because all stabilizers are based on X or Z, but not both, transversal for Clifford gates (X, Y, Z, S, CNOT)

State Preparation



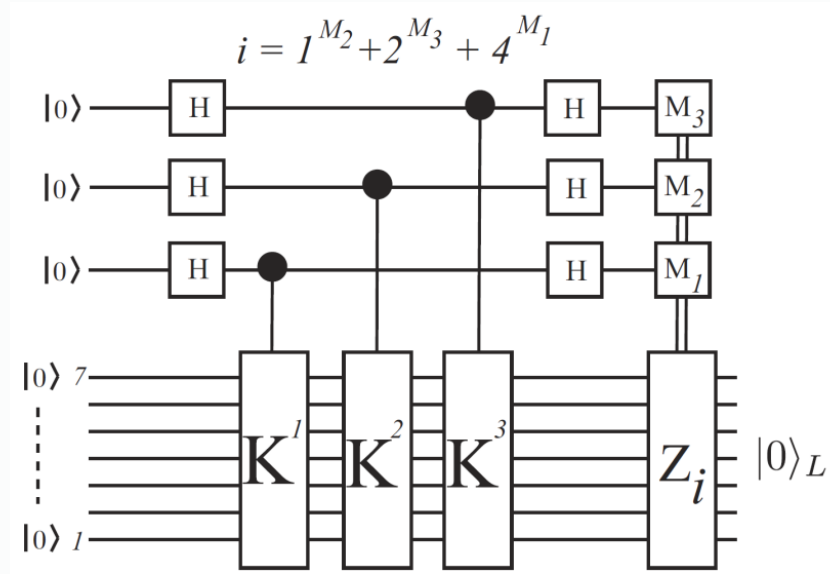
Project arbitrary state into eigenstate of Hermitian U

$$|\psi\rangle_F = \frac{1}{2}(|\psi\rangle_I + U\,|\psi\rangle_I)\,|0\rangle + \frac{1}{2}(|\psi\rangle_I - U\,|\psi\rangle_I)\,|1\rangle$$

After measurement…

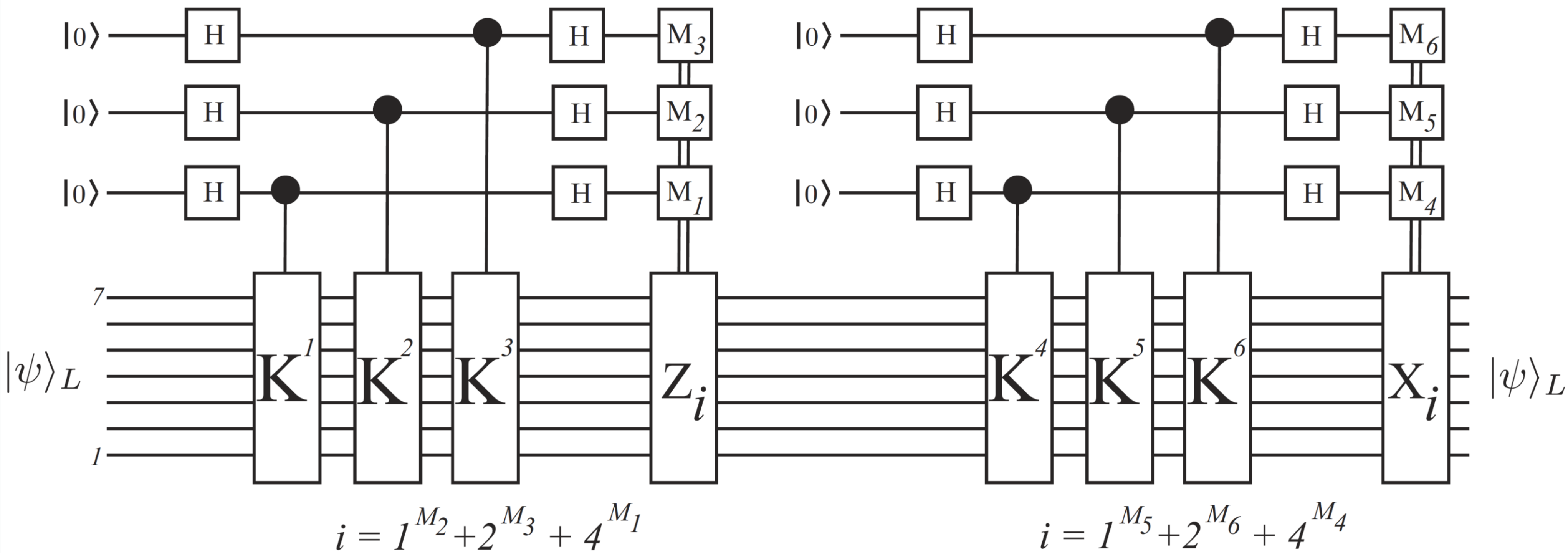
$|\psi\rangle_F = |\psi\rangle_I + U\,|\psi\rangle_I$ *or* $|\psi\rangle_F = |\psi\rangle_I - U\,|\psi\rangle_I$

+1 eigenstate *-1 eigenstate*



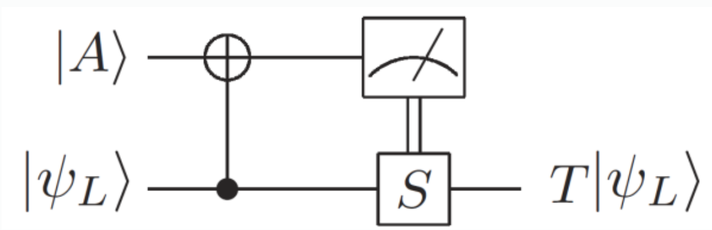
Z gate transforms -1 states to +1 states
No need to project to operators 4, 5, 6 because
0 is already an eigenstate

Error Correction



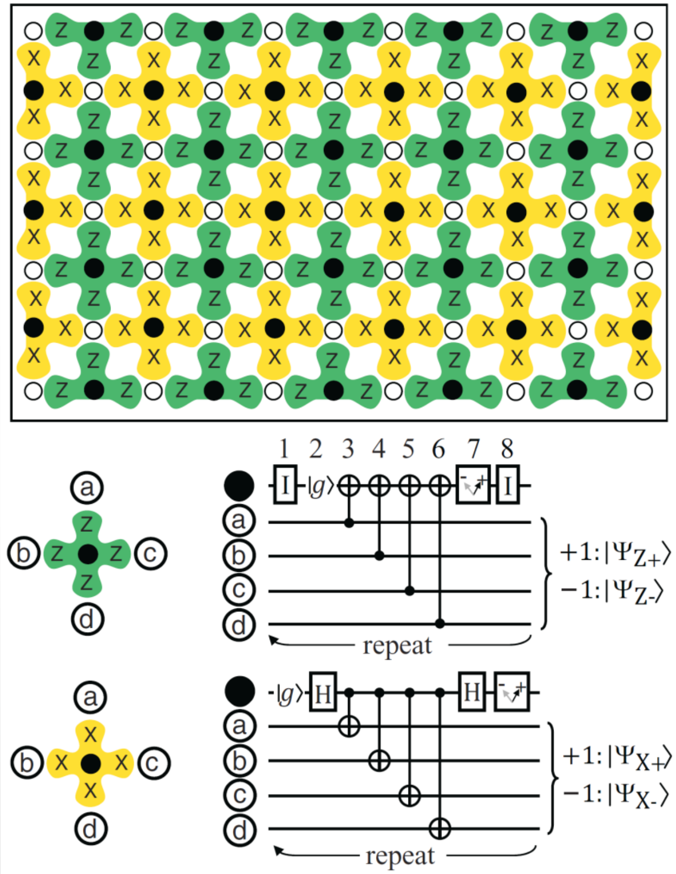
Universal Gates

- Steane code is transversal for Cliffort (X, Y, Z, S, CNOT), but universal QC also requires T gate
- Performing T (or Toffoli) is possible, but requires multiple single-qubit and two-qubit operators
  - 2-qubit operators can propagate errors, from single-qubit (correctable) to multi-qubit
- Can perform by "magic state" preparation



Fault Tolerance

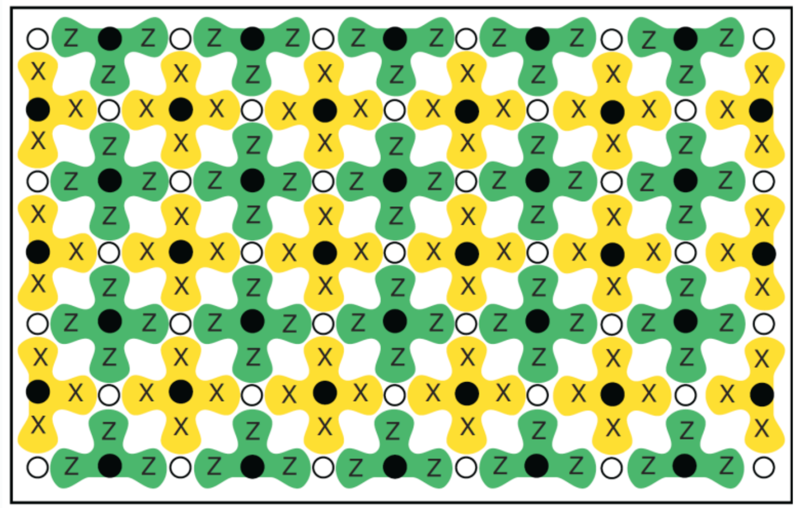
- More complicated than it appears…
- Error correction circuits are also subject to errors
  - Preparation of ancilla, magic states
  - Application of gates, measurement
  - "Distillation" of pure/low-error states
- More about this later



Surface Code

Surface Code

- Based stabilizers and repeated measurements in both the Z and X bases
- Qubits classified as "data" or "measurement"
- Requirements:
  - All qubits must allow single-qubit rotations and CNOT between nearest neighbors
  - For Hadamard, must be able to SWAP state with neighbors
  - Measurement in the Z basis

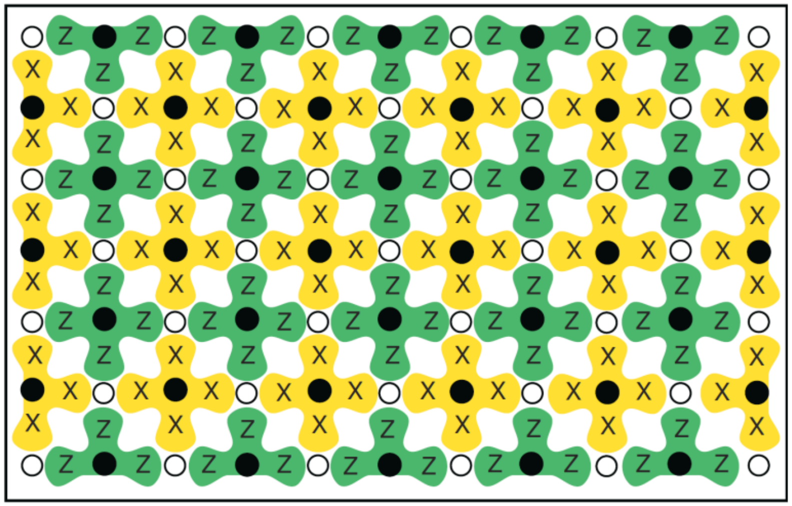


Measurement Qubits

- "measure-Z" qubits (green)
  "measure-X" qubits (yellow)

- measure-Z qubit forces its neighbors (a, b, c, d) into an eigenvalue of $Z_aZ_bZ_cZ_d$

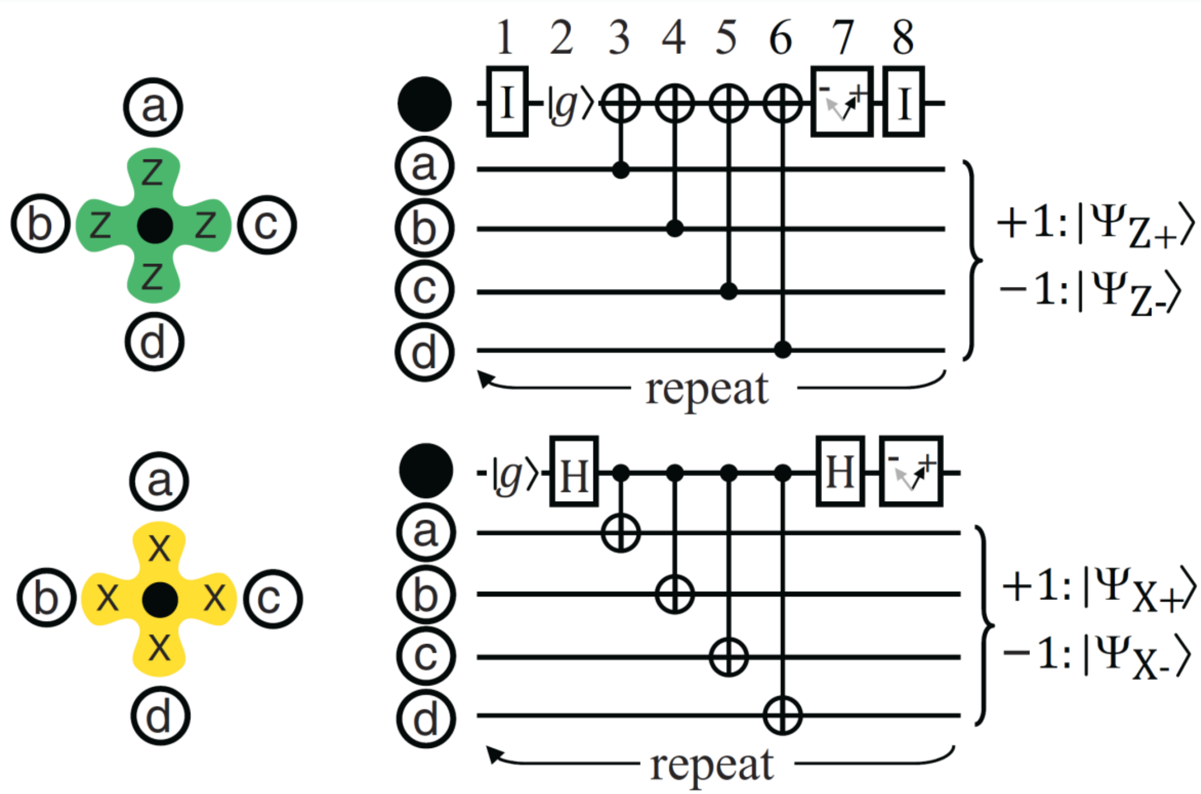
- measure-X qubit forces its neighbors (a, b, c, d) into an eigenvalue of $X_aX_bX_cX_d$



Stabilizers

Eigenvalue	$\hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d$	$\hat{X}_a\hat{X}_b\hat{X}_c\hat{X}_d$
$+1$	$\|gggg\rangle$	$\|++++\rangle$
	$\|ggee\rangle$	$\|++--\rangle$
	$\|geeg\rangle$	$\|+--+\rangle$
	$\|eegg\rangle$	$\|--++\rangle$
	$\|egge\rangle$	$\|-++-\rangle$
	$\|gege\rangle$	$\|+-+-\rangle$
	$\|egeg\rangle$	$\|-+-+\rangle$
	$\|eeee\rangle$	$\|----\rangle$
-1	$\|ggge\rangle$	$\|+++-\rangle$
	$\|ggeg\rangle$	$\|++-+\rangle$
	$\|gegg\rangle$	$\|+-++\rangle$
	$\|eggg\rangle$	$\|-+++\rangle$
	$\|geee\rangle$	$\|+---\rangle$
	$\|egee\rangle$	$\|-+--\rangle$
	$\|eege\rangle$	$\|--+-\rangle$
	$\|eeeg\rangle$	$\|---+\rangle$

Measurement Cycle



Error Detection

- Error causes changes in the measurement outcomes
- Does not try to correct (e.g. by applying X or Z)
  - Those operations are error-prone
- In classical software, tracks the errors in a qubit and adjusts subsequent measurement results (classically)
  - Later errors can "undo" the adjustment

Error Detection

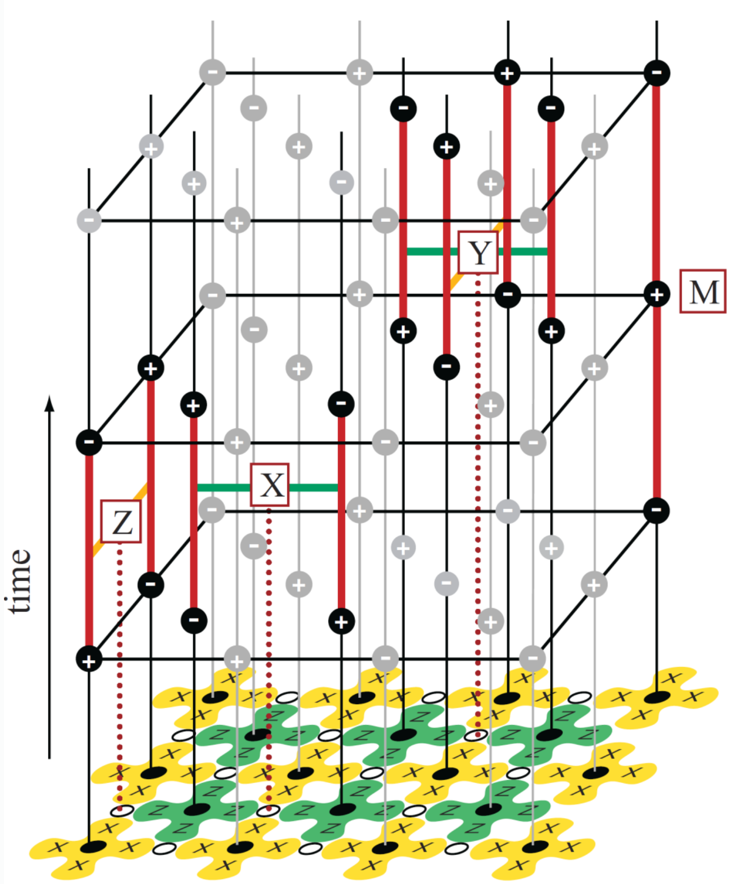
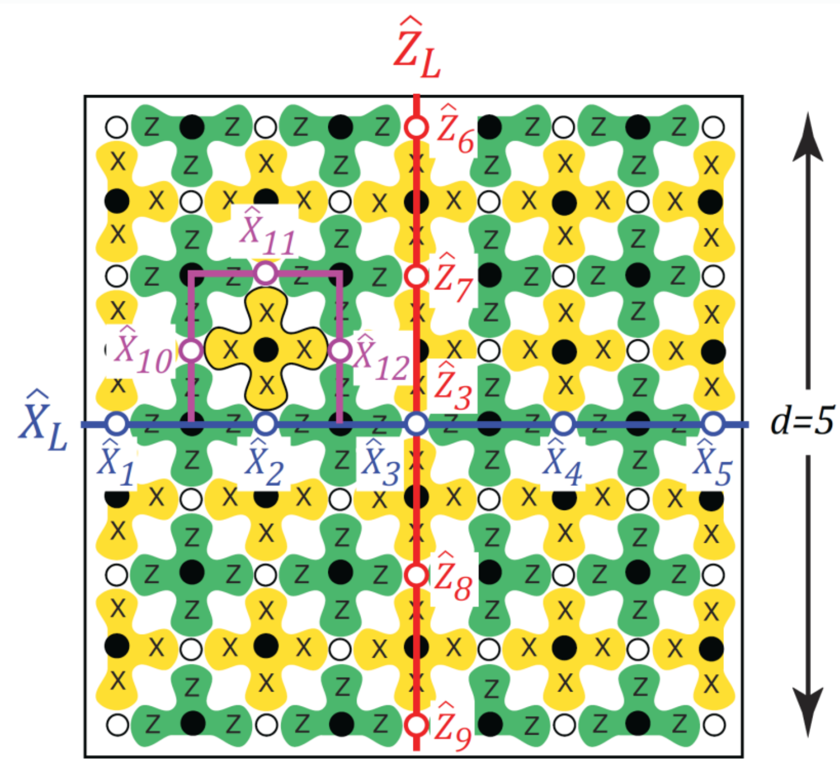


FIG. 2. (Color online) Schematic evolution of measurement outcomes (solid circles with ± signs), over a segment of the 2D array. Time progresses moving up from the array at the bottom of the figure, with measurement steps occurring in each horizontal plane. Vertical heavy red (gray) lines connect time steps in which a measurement outcome has changed, with the spatial correlation indicating an $\hat{X}$ bit-flip error, a $\hat{Z}$ phase-flip error, a $\hat{Y} = \hat{Z}\hat{X}$ error, and temporal correlation a measurement (M) error, which is sequential in time.

Logical Operations



Missing measurement qubits (e.g., on boundaries) introduce additional degrees of freedom. Ex: Diagram has 41 data qubits and only 40 stabilizers.

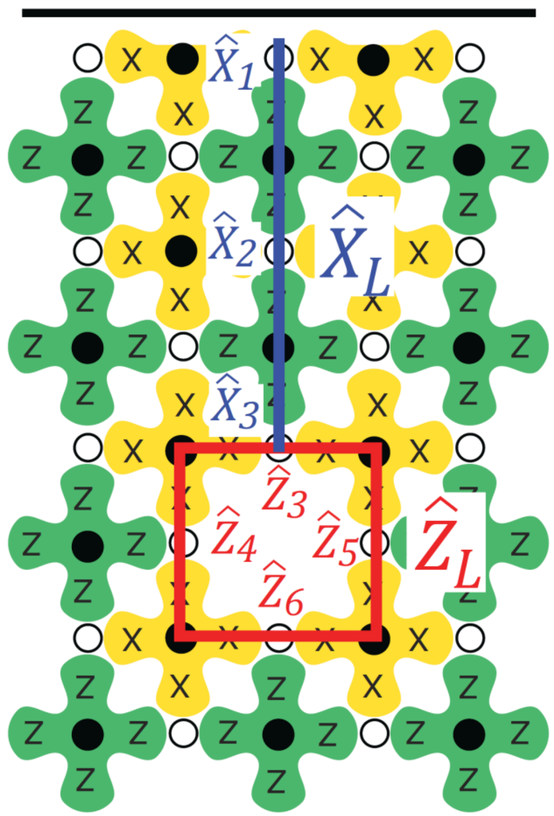
Can this array be viewed as a single logical qubit?

Applying X at ALL of the blue locations will alter the state of the array, but measurement results will remain the same. It has the affect applying a logical X to the logical qubit.

Likewise for Z operator applied along red line.

Large arrays are desired for low logical error rates. (More on this later.) How can we increase the number of logical qubits within an array?

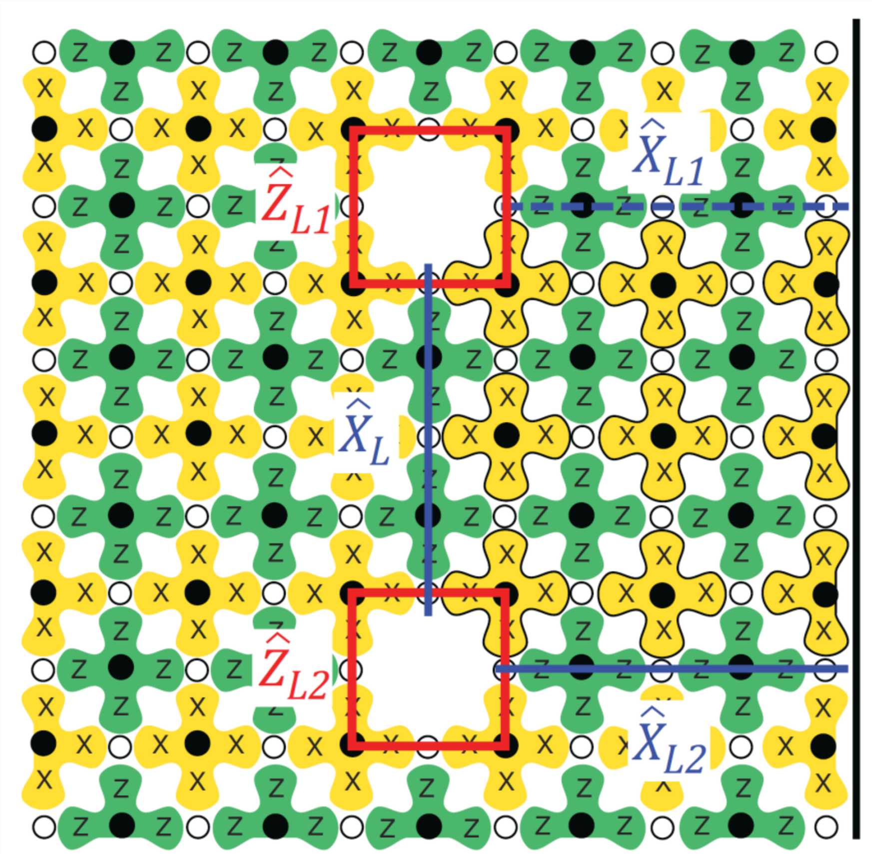
Logical Operations



Create "holes" (defects) to generate additional degrees of freedom. Just "turn off" one or more measurement bits.

Figure shows a "single Z-cut" qubit.

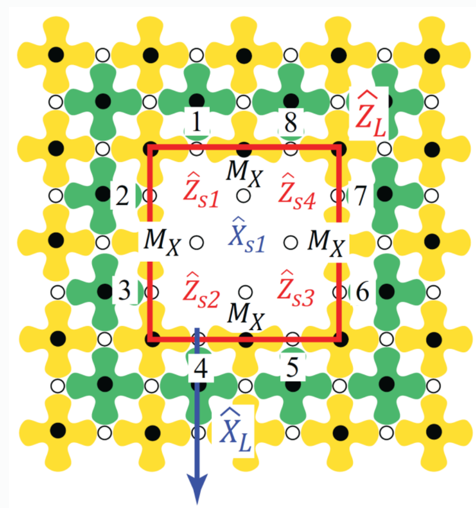
Logical Operations



A double Z-cut logical qubit.

Removes the requirement to be close to an X boundary.

Making larger holes allows larger distance (d), or better error tolerance.



Logical Operations

- Not enough time… (or expertise!)

- Hadamard, S, T
- Moving logical qubits
- Braiding -- moving logical qubits in relation to each other
  - Required for CNOT

How many physical qubits?

- Both stabilizer and surface codes need large numbers of physical qubits to reduce error rates to a desired logical error rate
- Multi-level codes
  - Level-2 encoding as a collection of Level-1 logical qubits
  - Increase levels until error rate is reached (e.g., 10^{-14} for a reasonable implementation of Shor's algorithm)
- Surface codes
  - Increasing d (dimension of array) improves logical error rate
- On the order of **thousands** of physical qubits per logical qubit