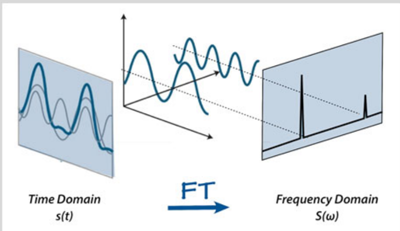
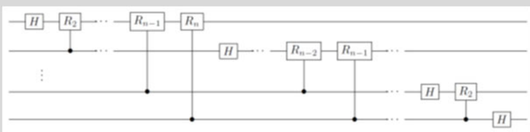




Quantum Fourier Transform

ECE 592/CSC 591 – Fall 2018

NC STATE UNIVERSITY Electrical & Computer Engineering

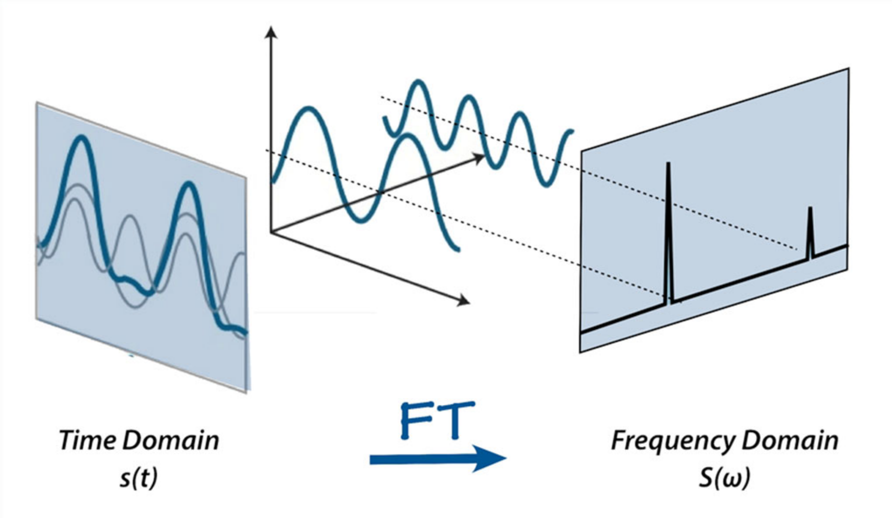
@NCStateECE

QFT

- Quantum version of FFT
- Building block of other quantum algorithms
- Transforms coefficients of n-qubit superposition into "frequency domain" – useful in period finding

- N-dimensional (n-qubit) QFT: $O(\log^2 N)$
  - Classical FFT: $O(N \log N)$
  - Exponential speedup

Review: Fourier Transform



http://mriquestions.com/fourier-transform-ft.html

DFT and FFT

DFT operates on a discrete complex-valued function a(x), and produces a new function A(x):

$$A(x) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a(k) e^{2\pi i \frac{kx}{N}}$$

FFT is an efficient implementation of DFT when N is a power of 2.

FFT

Let $\omega_{(n)}$ represent the Nth root of unity, $\omega_{(n)} = e^{\frac{2\pi i}{N}}$.

FFT can be expressed as a matrix transformation

$(F_N)_{jk} = \omega_{(n)}^{jk}$

Example, for $N = 4$, F_4 is a 4×4 matrix:

$$F_4 = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega_{(2)}^0 & \omega_{(2)}^0 & \omega_{(2)}^0 & \omega_{(2)}^0 \\ \omega_{(2)}^0 & \omega_{(2)}^1 & \omega_{(2)}^2 & \omega_{(2)}^3 \\ \omega_{(2)}^0 & \omega_{(2)}^2 & \omega_{(2)}^4 & \omega_{(2)}^6 \\ \omega_{(2)}^0 & \omega_{(2)}^3 & \omega_{(2)}^6 & \omega_{(2)}^9 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Implementation efficiency comes from recursive nature of FFT – more about this later.

QFT uses the same matrix

$$\sum_x a(x)\,|x\rangle \rightarrow \sum_x A(x)\,|x\rangle$$

Function is expressed as coefficient of n-qubit state
No output register – change in state.

Same matrix, normalization makes it unitary

$$\mathrm{QFT}_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ a(3) \end{pmatrix} = \begin{pmatrix} A(0) \\ A(1) \\ A(2) \\ A(3) \end{pmatrix}$$

Effects of QFT

$$\sum_x a(x)\,|x\rangle \rightarrow \sum_x A(x)\,|x\rangle$$

If a_x is periodic with period r, which is a power of 2, then A_x will be zero except when x is a multiple of $\frac{N}{r}$.

When measured, resulting state will be a random multiple of $\frac{N}{r}$.

If period is not a power of 2, then states near multiples of $\frac{N}{r}$ will be measured with high probability.

With one qubit…

- One-qubit QFT = Hadamard gate.

$$\omega_{(1)} = e^{\frac{2\pi i}{2}} = -1$$

$$\mathrm{QFT}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Implementation

- To the "chalkboard..."

Effect

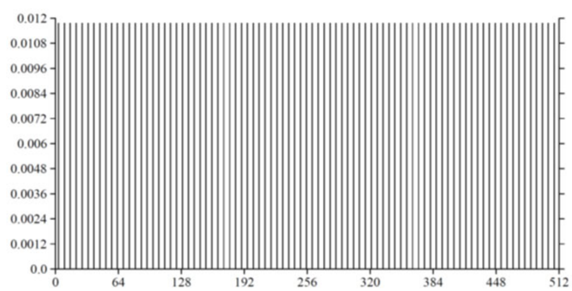


Fig. 2. Probabilities for measuring x when measuring the state $C\sum_{x\in X}|x,8\rangle$ obtained in Step 2, where $X = \{x|211^x \bmod 21 = 8\}\}$

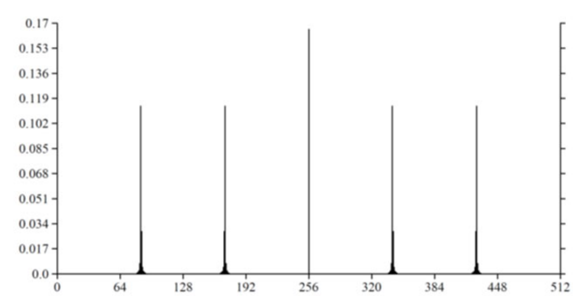


Fig. 3. Probability distribution of the quantum state after Fourier Transformation.