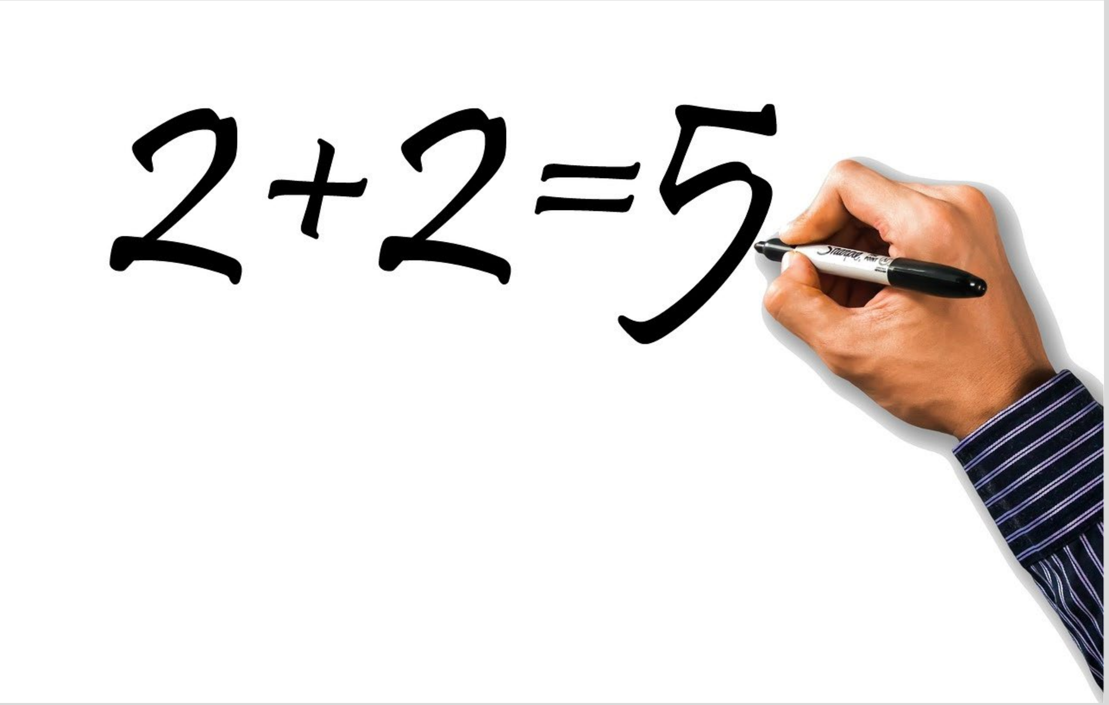

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Quantum Error Correction: Part 1, Stabilizer Codes

ECE 592/CSC 591 – Fall 2019

J. Roffe, arXiv:1907.11157v1
S. Devitt, et al., arXiv:0905.2794v4

Errors

- Sources of errors
  - Control errors -- gates that are incorrectly applied
    - Usually based on faulty knowledge of the physical system characteristics
  - Environmental errors
  - Initialization, measurement, loss, leakage (into other energy states)

- What they look like
  - Random qubit flip (X) and phase change (Z)

Coherent Errors

Coherent error = rotate to another point on Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$U(\delta\theta,\delta\phi)|\psi\rangle = \cos\frac{\theta+\delta\theta}{2}|0\rangle + e^{i(\phi+\delta\phi)}\sin\frac{\theta+\delta\theta}{2}|1\rangle$$

$$U(\delta\theta,\delta\phi)|\psi\rangle = \alpha_I \mathbb{1}|\psi\rangle + \alpha_X X|\psi\rangle + \alpha_Z Z|\psi\rangle + \alpha_{XZ} XZ|\psi\rangle$$

Any coherent error can be viewed as combination of X and/or Z.

Challenges

No cloning

Can’t replicate qubits to gain redundancy

Both X (flip) and Z (phase) errors

Classical bits only exhibit flip errors

Quantum error codes have to handle both types

Measurement = wavefunction collapse

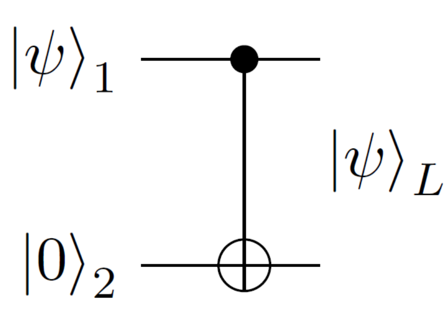
Can’t directly measure qubits: use projective stabilizer measurements…

Quantum Error Codes

Instead of replicating bits, embed into larger Hilbert Space.

Example: **two-qubit code**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{two-qubit encoder}} |\psi\rangle_L = \alpha|00\rangle + \beta|11\rangle = \alpha|0\rangle_L + \beta|1\rangle_L$$



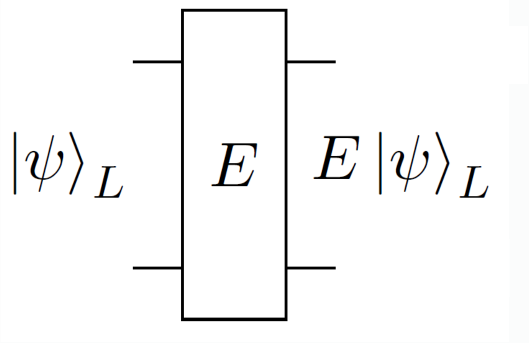
$$\mathcal{H}_4 = \text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

$$|\psi\rangle_L \in \mathcal{C} = \text{span}\{|00\rangle, |11\rangle\} \subset \mathcal{H}_4$$

Warning: numbering scheme!

Quantum Error Codes

Suppose a bit flip error (X_1) happens on qubit 1…



$$X_1 |\psi\rangle_L = \alpha |10\rangle + \beta |01\rangle$$

$$X_1 |\psi\rangle_L \in \mathcal{F} \subset \mathcal{H}_4$$

Flip error takes state into a different subset of the Hilbert space.
F is orthogonal to C, so good states can be distinguished from error states.

Warning: numbering scheme!

Syndrome Extraction

How do we distinguish between ***C*** and ***F*** subspaces without measuring?

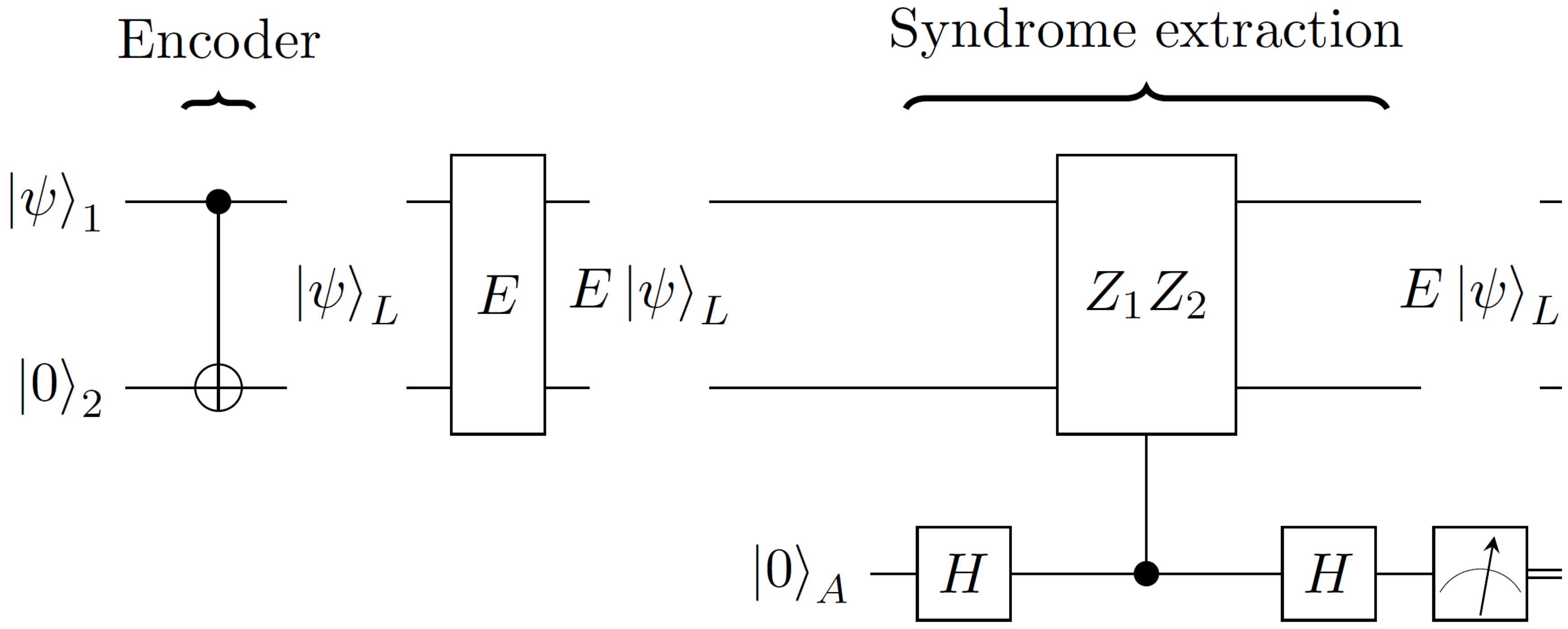
Z_1Z_2 is a stabilizing operator (**stabilizer**) for $|\psi\rangle_L$ -- eigenvalue is +1.

$$Z_1Z_2|\psi\rangle_L = Z_1Z_2(\alpha|00\rangle + \beta|11\rangle) = (\alpha|00\rangle + \beta|11\rangle) = (+1)|\psi\rangle_L$$

If an error occurs...

$$Z_1Z_2X_1|\psi\rangle_L = Z_1Z_2(\alpha|10\rangle + \beta|01\rangle) = (-\alpha|10\rangle - \beta|01\rangle) = (-1)X_1|\psi\rangle_L$$

Syndrome Extraction



Error	Syndrome, S
$I_1 I_2$	0
$X_1 I_2$	1
$I_1 X_2$	1
$X_1 X_2$	0

1, if $E = \{X_1, X_2\}$

0, if $E = \{\mathbb{1}, X_1 X_2\}$

$$E|\psi\rangle_L |0\rangle_A \xrightarrow{\text{syndrome extraction}} \frac{1}{2}(\mathbb{1}_1\mathbb{1}_2 + Z_1 Z_2)E|\psi\rangle_L |0\rangle_A + \frac{1}{2}(\mathbb{1}_1\mathbb{1}_2 - Z_1 Z_2)E|\psi\rangle_L |1\rangle_A$$

So far…

Two-qubit code

- Detects bit flip error, but not phase error
- Not enough information to correct error

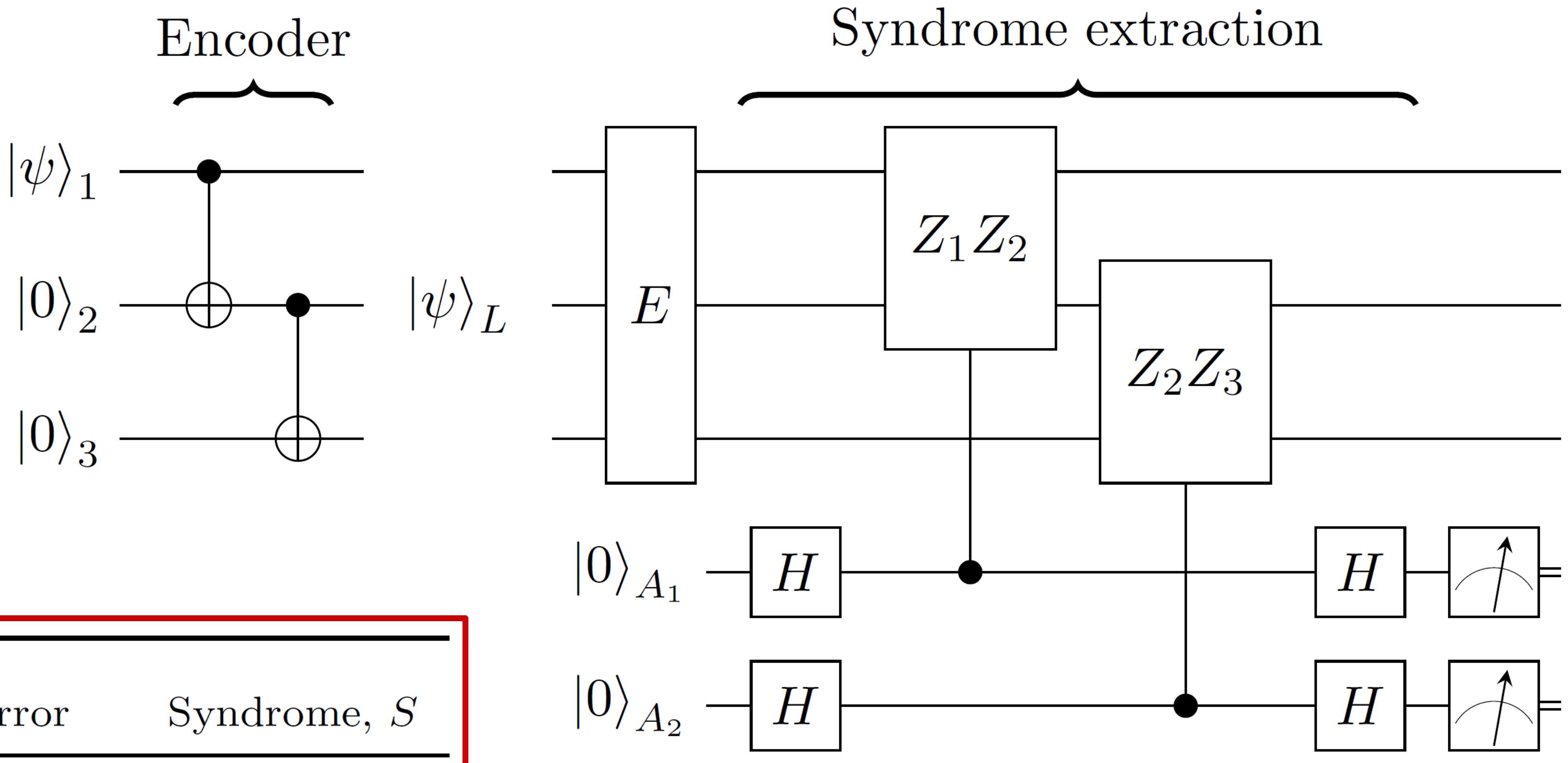
Stabilizer

- Leaves “good” state unchanged (+1 eigenvalue)
- Projects “bad” state into -1 eigenspace
- Projective measurement allows ancilla bit to detect the difference

3-qubit code

$$|0\rangle_L = |000\rangle$$

$$|1\rangle_L = |111\rangle$$



Error	Syndrome, S	Error	Syndrome, S
$I_1 I_2 I_3$	00	$X_1 X_2 I_3$	01
$X_1 I_2 I_3$	10	$I_1 X_2 X_3$	10
$I_1 X_2 I_3$	11	$X_1 I_2 X_3$	11
$I_1 I_2 X_3$	01	$X_1 X_2 X_3$	00

Corrects single bit flips, but not phase flips.

Generalized Error Codes

$$[\![n, k, d]\!]$$

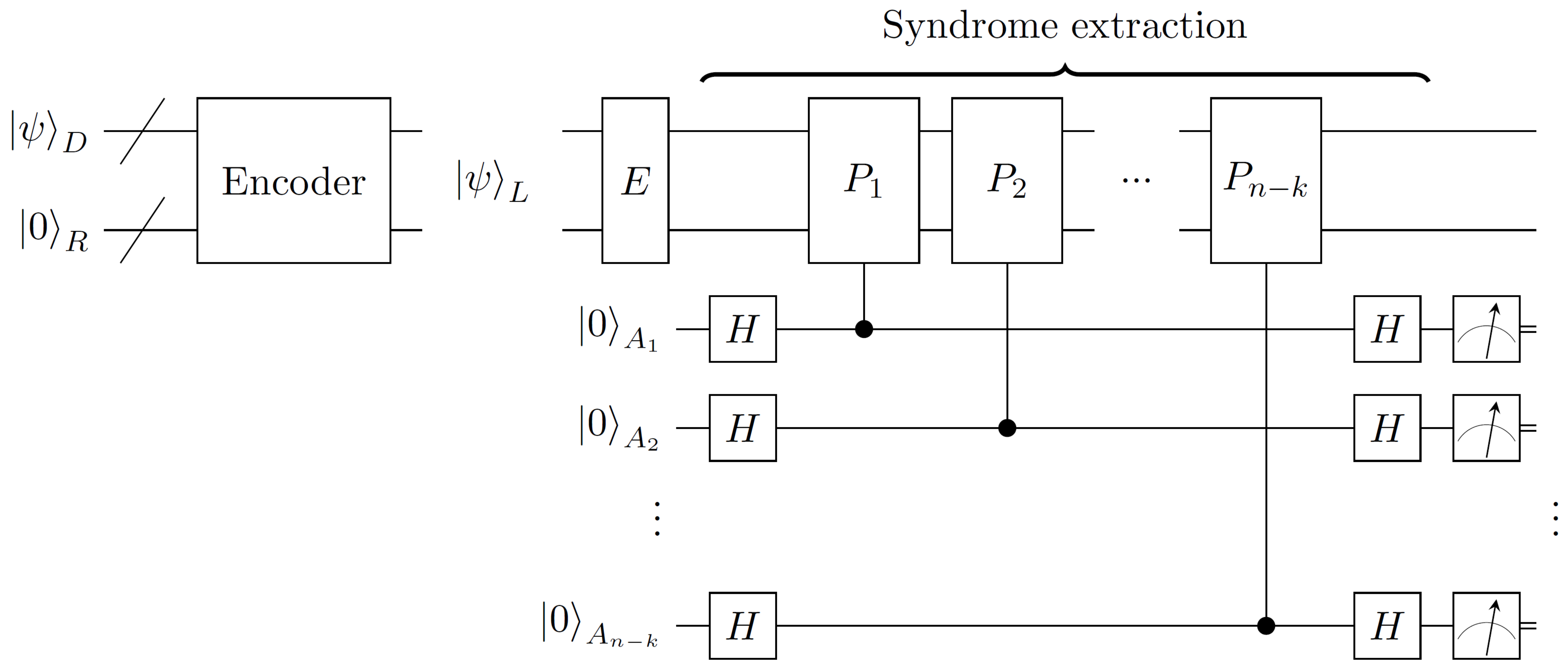
number of physical qubits

number of logical qubits

distance:
min # errors that transform one code word to another

can correct t errors, where

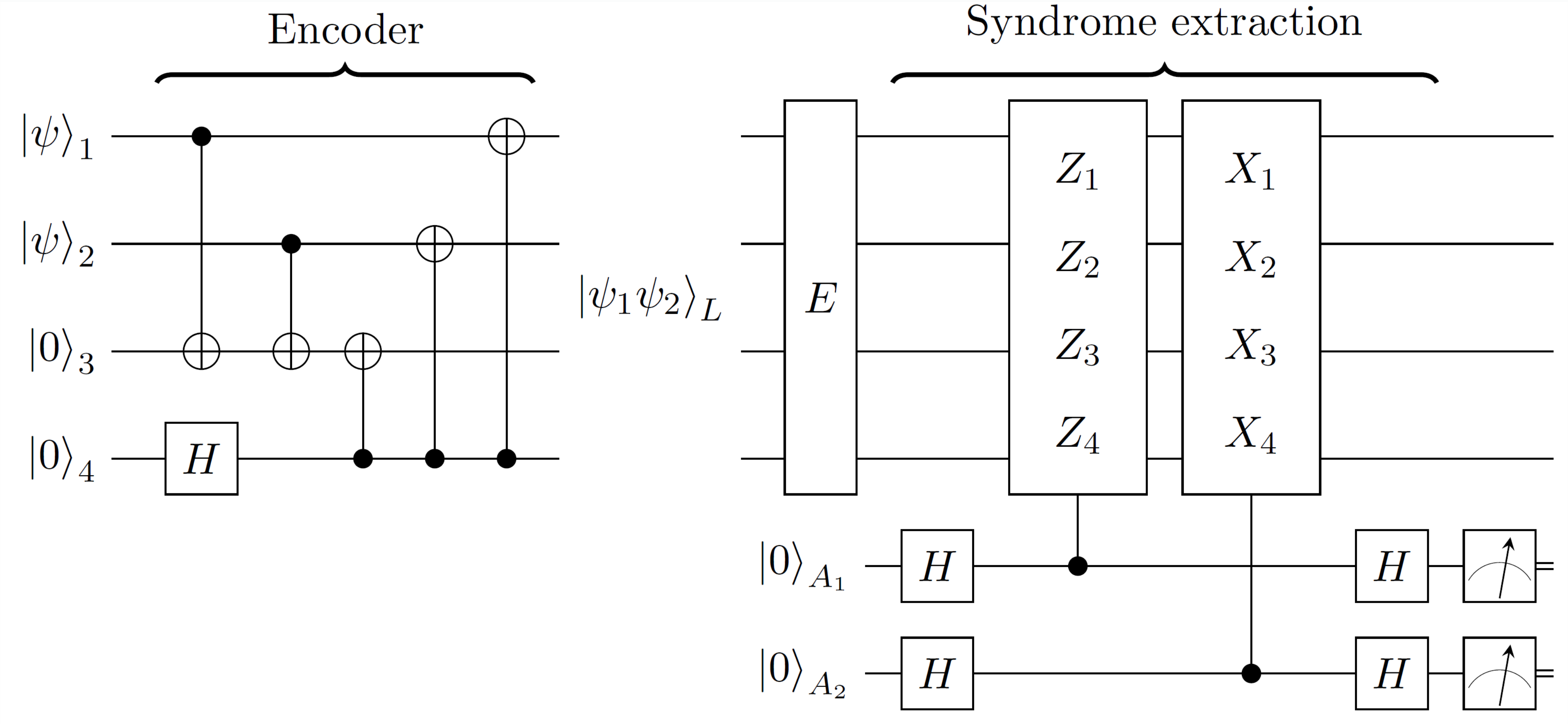
$$d = 2t + 1$$



Syndrome extraction
$|\psi\rangle_D$
$|0\rangle_R$
Encoder
$|\psi\rangle_L$
E
P_1
P_2
P_{n-k}
$|0\rangle_{A_1}$
$|0\rangle_{A_2}$
$|0\rangle_{A_{n-k}}$
H

[[4,2,2]] Code

Smallest stabilizer code that detects single-qubit flip and phase errors.



[[4,2,2]] Code

Smallest stabilizer code that detects single-qubit flip and phase errors.

$$|00\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

$$\mathcal{S}_{[[4,2,2]]} = \langle X_1X_2X_3X_4, Z_1Z_2Z_3Z_4\rangle$$

$$|01\rangle_L = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle)$$

$$|10\rangle_L = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)$$

$$|11\rangle_L = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

Error	Syndrome, S	Error	Syndrome, S	Error	Syndrome, S
X_1	10	Z_1	01	Y_1	11
X_2	10	Z_2	01	Y_2	11
X_3	10	Z_3	01	Y_3	11
X_4	10	Z_4	01	Y_4	11

[[4,2,2]] Logical Operators

$$|00\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

$$|01\rangle_L = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle)$$

$$|10\rangle_L = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)$$

$$|11\rangle_L = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle)$$

$$\bar{X}_1 = X_1 X_3$$
$$\bar{Z}_1 = Z_1 Z_4$$
$$\bar{X}_2 = X_2 X_3$$
$$\bar{Z}_2 = Z_2 Z_4$$

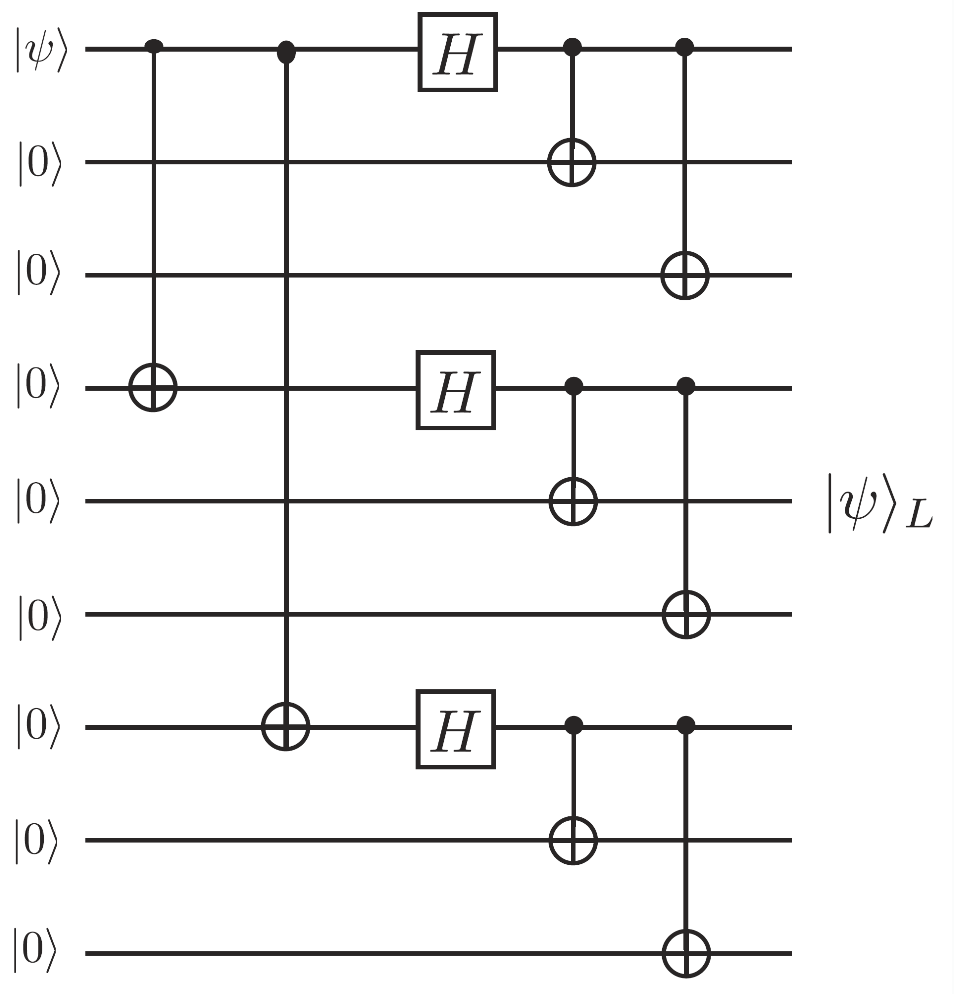
9-qubit Shor code

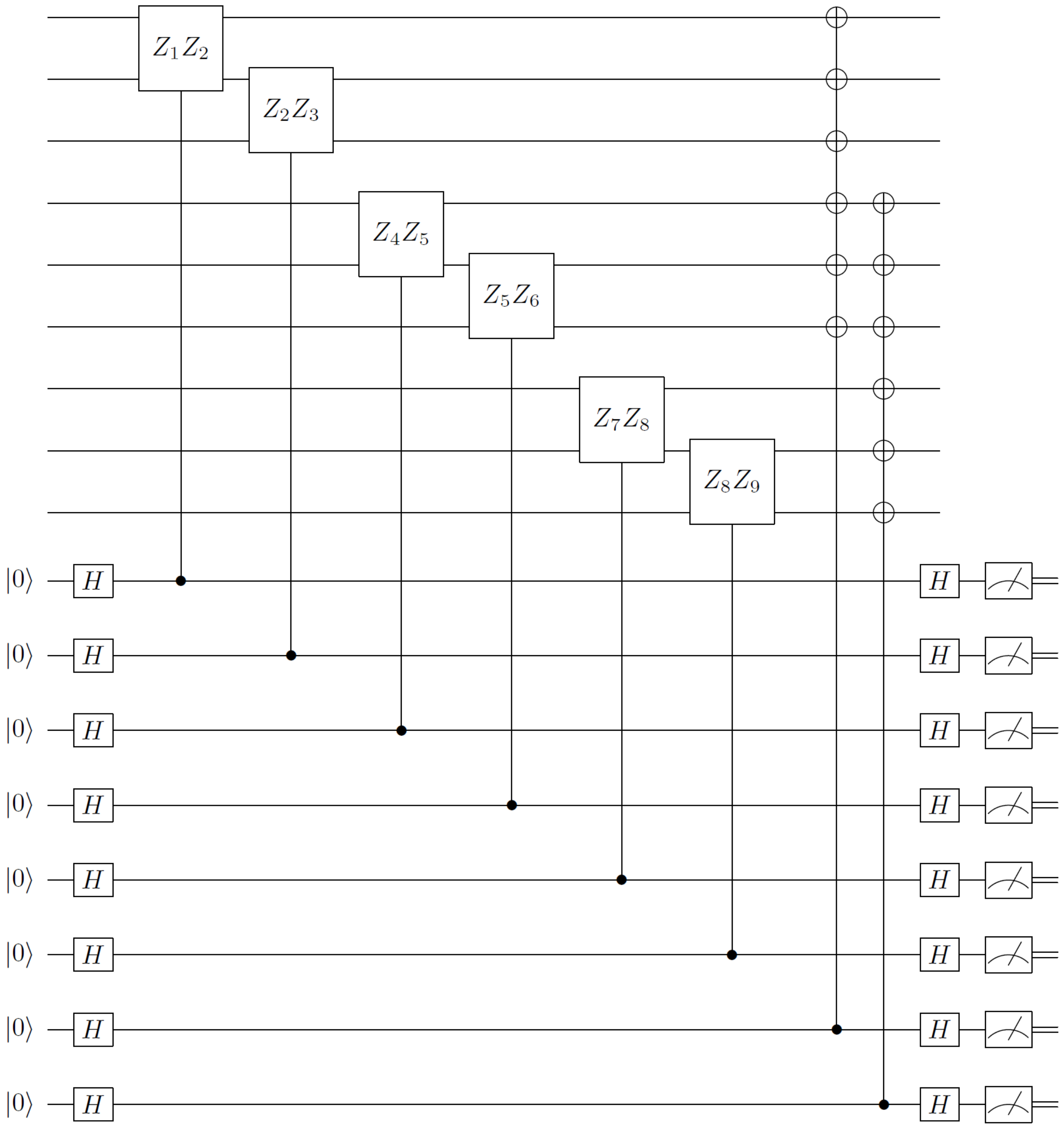
First complete quantum code (1995)

Corrects a single flip (X) or phase (Z) error, or both

$$|0\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Flip correction = same circuit as before on each group of three.





Stabilizers:

Z_1Z_2, Z_2Z_3

Z_4Z_5, Z_5Z_6

Z_7Z_8, Z_8Z_9

$X_1X_2X_3X_4X_5X_6$

$X_4X_5X_6X_7X_8X_9$

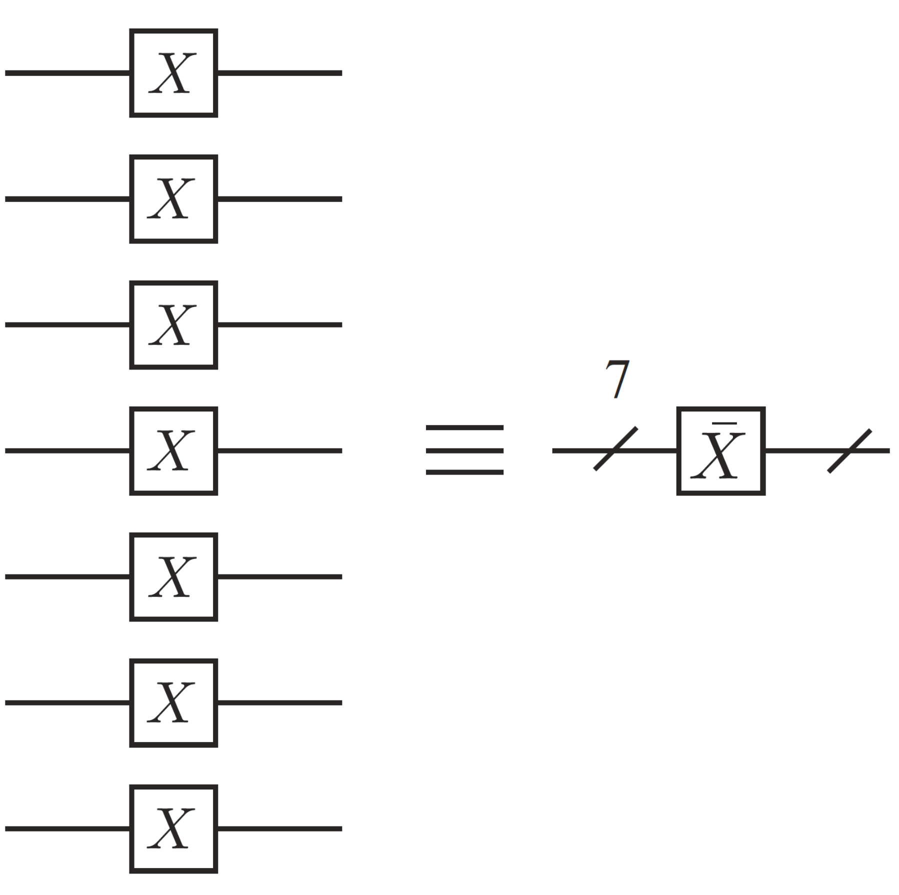
Summary of 9-qubit Shor Code

- Correct a single flip error in any of the nine qubits
  - Actually, can correct one qubit in all three groups, but this is considered a single-qubit code because not all multi-flip errors can be corrected.
- Correct a single phase error in any of the nine qubits
  - Correct with a Z on any qubit of the group
- If both X and Z error occurs, both will be corrected
  - Even if on the same qubit

Transversal Gates

Transversal gates

- If an operator is transversal, then applying the operator (gate) to an encoded state is a matter of applying the same (or equally simple) gate to each individual bit
- This means that computations can be just as efficient with encoded states



7-qubit Steane Code [[7, 1, 3]]

$$|0\rangle_L = \frac{1}{\sqrt{8}}(\,|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$$

$$|1\rangle_L = \frac{1}{\sqrt{8}}(\,|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle)$$

$$K^1 = IIIXXXX, \quad K^2 = XIXIXIX,$$
$$K^3 = IXXIIXX, \quad K^4 = IIIZZZZ$$
$$K^5 = ZIZIZIZ, \quad K^6 = IZZIIZZ.$$

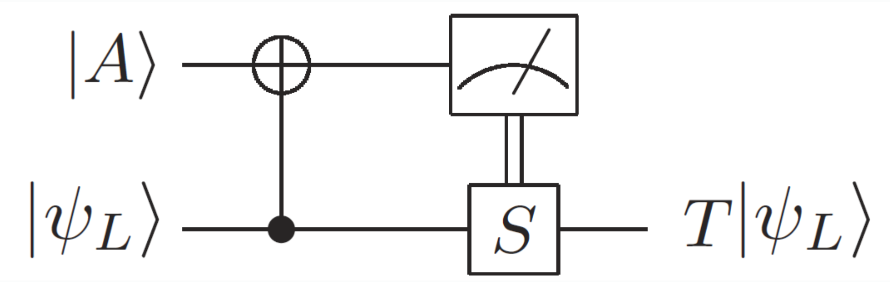
Stabilizers

[[7,1,3]] code
- 7 physical qubits
- 1 logical qubit
- distance 3 between states
- corrects (3-1)/2 = 1 error

Because all stabilizers are based on X or Z, but not both, transversal for Clifford gates (H, S, CNOT)

Universal Gates

- Steane code is transversal for X, Y, Z, Clifford (H, S, CNOT), but universal QC also requires T gate
- Performing T is possible, but requires multiple single-qubit and two-qubit operators
  - 2-qubit operators can propagate errors, from single-qubit (correctable) to multi-qubit
- Can perform by "magic state" preparation



Fault Tolerance

- More complicated than it appears…
- Error correction circuits are also subject to errors
  - Preparation of ancilla, magic states
  - Application of gates, measurement
  - "Distillation" of pure/low-error states

Threshold Theorem

Suppose each **physical** qubit experiences an X and/or Z error with probability p for each gate.

Given [[n,k,3]] code, will correct that single error.

So a logical error requires at least two physical errors: $p_{L1} = cp^2$

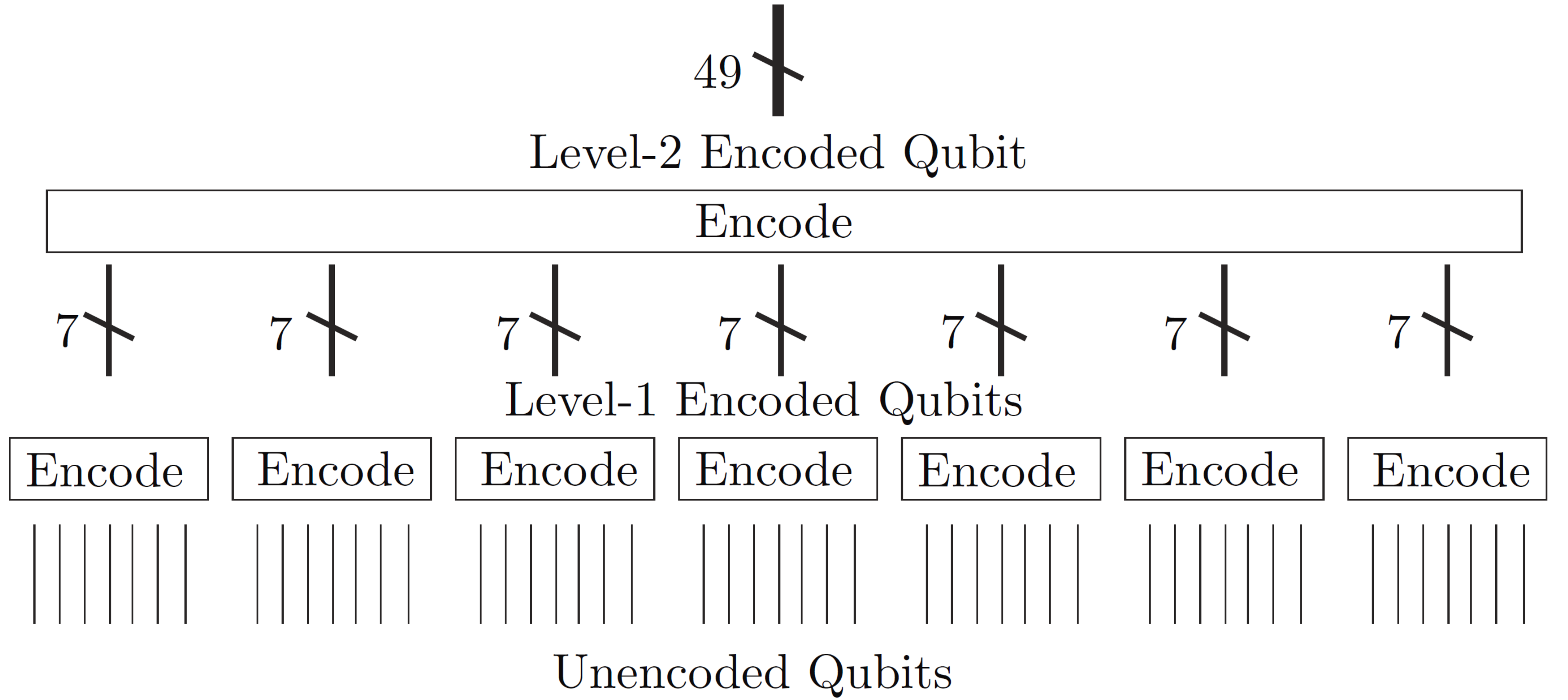
Encode those logical qubits in a level-2 [[n,k,3]] code: $p_{L2} = c(p_{L1})^2 = c^3p^4$

With g levels:

$$p_{Lg} = \frac{(cp)^{2^g}}{c}$$

c is upper bound on
the number of 2-error combinations
during correction cycle, logical gate,
and second correction cycle

Can reduce error rate arbitrarily low if $cp < 1$.


49
Level-2 Encoded Qubit
Encode
7
7
7
7
7
7
7
Level-1 Encoded Qubits
Encode
Encode
Encode
Encode
Encode
Encode
Encode
Unencoded Qubits