Properties of Linear Algebra Applicable to Quantum Computing Part II

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OUTLINE:

Mathematics and Physics Concepts Required to Describe a Quantum Computing System

- Introduce relevant mathematics applicable to quantum computing
- Introduce quantum mechanics axioms that describe observed physics
- Combine relevant mathematics with the physics of quantum mechanics to allow one to formulate
  - Design programming primitives that can aggregate into quantum computing algorithms
  - Program algorithms for implementation on quantum simulators and HW platforms
  - Configure and run these algorithms on quantum computing simulators and quantum computing hardware platforms

Roadmap For Designing a Gate Based Quantum Computer

- Assemble the mathematical language to describe the quantum mechanical dynamics being harnessed in order to build a quantum computing hardware platform
- The underlying quantum mechanical dynamics of the physical system is initialized by defining qubits in some initial state based on the axioms of quantum mechanics
- These qubits evolve through a sequence of applied unitary operations and projective measurements (called gates) that manipulate the states of the qubits
- Sequences of gates are assembled into a circuit that represents a set of instructions that model the problem being implemented on a quantum computer
- Circuit instructions are compiled and delivered to the qubits in the quantum computer as a set of microwave control pulses
- These microwave pulses implement the desired unitary quantum mechanical state-transformations and/or measurements by steering or evolving these qubits from an initialized state through final measurement
- The final measurement extracts classical information in the form of bit strings, which encode the outcome of projective measurements of the qubits in a particular measurement basis according to the axioms of quantum mechanics and the mathematics of linear algebra

Properties of Linear Algebra Applicable to Quantum Computing Part II

Rotation Group

- Recall from the Part 1 lecture
- The "defining" representation of the rotation group is three dimensional, but the simplest nontrivial irreducible representation is two dimensional
- In this case there is a unique two-dimensional irreducible representation, up to a unitary change of basis
- The generators of this rotation group are

Pauli X — X — $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$

Pauli Y — Y — $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$

Pauli Z — Z — $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$

} Pauli Matrices*

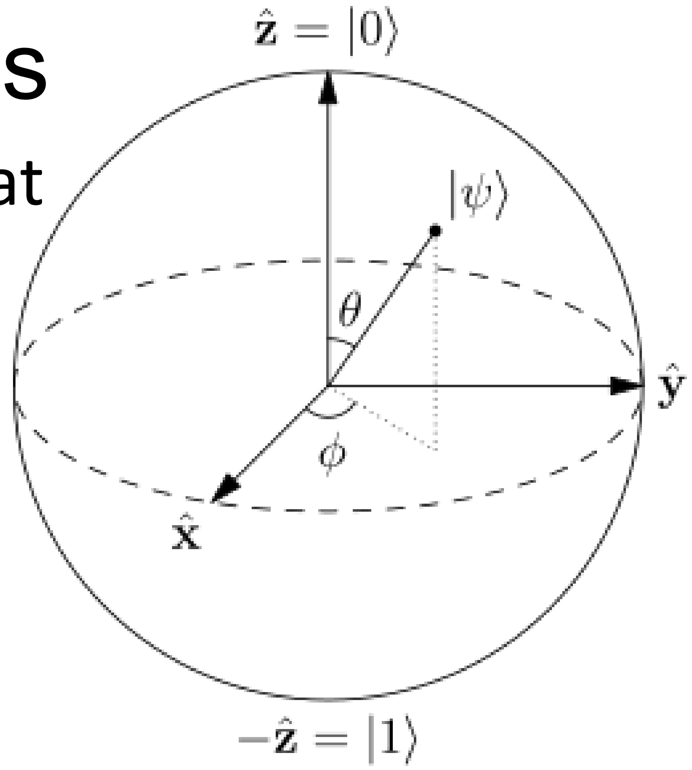
* These Pauli Matrices have a special relationship in physics to particles that carry a property known as "spin"

Mathematical Representation of Many Different Basis States

- Represent combination of "0"s and "1"s in a way that many different values can be expressed
- Define $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
- Can re-write $|a\rangle = \alpha|0\rangle + \beta|1\rangle$ as $|\alpha|^2 + |\beta|^2 = 1$

$$|a\rangle = e^{i\gamma}(\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle)$$

- This representation is visualized by states that lie of the surface of a Bloch sphere



Bloch Sphere

Figure from Wikipedia Bloch Sphere https://en.wikipedia.org/wiki/Bloch_sphere

- The Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system

Matrices as Rotations Acting on Qubits

- Matrices describe the rotations that takes a qubit from an initial state to a transformed state

- These rotations that operate on a qubit are labelled as “gates”

- Because qubit states can be represented as points on a sphere, reversible one-qubit gates can be thought of as rotations of the Bloch sphere. (quantum gates are often called “rotations”)

Construct Rotation Matrices From Bra and Ket Vectors

- **The matrix representation of the expression** $\sum_i |input_i >< output_i|$

$$I = |0 >< 0| + |1 >< 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = |0 >< 1| + |1 >< 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Z = |0 >< 0| - |1 >< 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y = iXZ = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} [(|0 > +|1 >) < 0| + (|0 > -|1 >) < 1|] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Building Tensor Product from Matrices

- Let A and B be represented by the following matrices

$$A = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \qquad B = \begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix}$$

$$A \otimes B = \begin{bmatrix} a\begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix} & b\begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix} \\ c\begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix} & d\begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix} \end{bmatrix}$$

General Statement - Outer Products

- Any matrix can be written purely in terms of its outer products (example)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

- This is a useful formulation to express linear transformations
- Select an original set of basis states (orthogonal) and express in this outer product representation
- Can directly read the effect of the unitary on the basis stated

Properties of Outer Products

- Given vectors **U, V**, and **W** and a scalar c

$$(\mathbf{U} \otimes \mathbf{V})^{T} = (\mathbf{V} \otimes \mathbf{U})$$

$$(\mathbf{V}+\mathbf{W}) \otimes \mathbf{U} = \mathbf{V} \otimes \mathbf{U} + \mathbf{W} \otimes \mathbf{U}$$

$$\mathbf{U} \otimes (\mathbf{V} + \mathbf{W}) = \mathbf{U} \otimes \mathbf{V} + \mathbf{U} \otimes \mathbf{W}$$

$$c\,(\mathbf{V} \otimes \mathbf{W}) = (c\,\mathbf{V}) \otimes \mathbf{W} = \mathbf{V} \otimes (c\,\mathbf{W})$$

NOTE: The outer product of tensors also satisfies an additional associativity property

$$\mathbf{U} \otimes (\mathbf{V} \otimes \mathbf{W}) = (\mathbf{U} \otimes \mathbf{V}) \otimes \mathbf{W}$$

Properties of Complex Matrices

- **Hermitian Matrix** – A matrix is defined to be a Hermitian matrix if it is a complex square matrix that is equal to its own conjugate transpose—(the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j)
- **Unitary matrix -** a complex square matrix whose adjoint equals its inverse
  - the product of $U^{\dagger}$ and the matrix U is the identity matrix
  - Note: a complex square matrix U is unitary if its conjugate transpose is also its inverse U^{-1})

$$U^{\dagger}U = U^{-1}U = I$$

States

- A pure quantum state can be represented by a ray in a Hilbert space over the complex numbers
- Pure states are also known as state vectors or wave functions
- Mixed states are represented by density matrices, which are positive semidefinite operators that act on Hilbert spaces (re-visit density matrices when discussing noise in quantum computer systems)
- A mixed quantum state corresponds to a probabilistic mixture of pure states; however, different distributions of pure states can generate equivalent (i.e., physically indistinguishable) mixed states

State Transformations

- Outer products are a useful mechanism for writing matrices, especially unitaries because they capture state transformations
- Pick an orthogonal set of states (ex pair of $|0\rangle$ and $|1\rangle$) and define a set of states $\{|u_{00}\rangle, |u_{01}\rangle, |u_{10}\rangle, |u_{11}\ \}$ to which to which the unitary rotates the original set of orthogonal states

$$U=|u_{00}\rangle\langle 00|+|u_{01}\rangle\langle 01+|u_{10}\rangle\langle 10| + |u_{11}\rangle\langle 11|$$

- This expression is not unique
- This is a general expression that can be constructed for every possible set of orthogonal input states

State Transformations and Concept of a Phase

- There will be at least one set of orthogonal input states that will take the form of eigenstates of the matrix

$$A = \sum_j \alpha_j \, |e_j >< e_j|$$

where $\alpha_j = \sum_j .\exp(i\, e_j)$

- The unitary maps each state of the basis $|e_j> \rightarrow \exp(i\, e_j)|e_j>$
- The transformed state is also a valid basis
  - Implies that the exponential terms must be complex number of magnitude 1
  - The e_j are real numbers
- This formalism also introduces a relative phase when a superposition of these states are combined

Hermitian Matrices and Unitaries

- Hermitian matrices have well defined eigenvalues and eigenstates
- They can be written in the same form as the unitary matrix "A"

$$\mathrm{H} = \sum_j h_j \, |h_j >< h_j|$$

- Hermitian matrices have the property that $H = H^\dagger$
- This requirement forces the eigenvalues and eigenvectors to have specific properties

Hermitian Matrices and Unitaries

- Using the property $|h_j>^{\dagger} = <h_j|$ examine the inner product

$$(|h_j><h_j|)^{\dagger} = (<h_j|^{\dagger})(|h_j>^{\dagger}) = |h_j><h_j|$$

- For this to be true the eigenvalues h_j of a Hermitian matrix must be real

Relationship between Unitary and Hermitian

- A unitary matrix (U) has complex exponentials of real numbers for eigenvalues
- Hermitian matrix (H) must have real numbers for eigenvalues
- Based on above 2 statements it is possible to define a Hermitian matrix from every unitary
- The eigenvalues can be related through exponentiation using the definition for exponentiation of a matrix*

$$U=\exp(iH)$$

* An entire family of unitaries can be constructed for each Hermitian

Summary

- A pure quantum state can be represented by a ray in a Hilbert space over the complex numbers
- Pure states are also known as state vectors or wave functions
- A mixed quantum state corresponds to a probabilistic mixture of pure states
- Different distributions of pure states can generate equivalent (i.e., physically indistinguishable) mixed states
- Mixed states are represented by density matrices (next tutorial)
- Density matrices are positive semidefinite operators that act on Hilbert spaces

Questions